J. Appl. Maths Mechs Vol. 56, No. 6, pp. 843–853, 1992
Printed in Great Britain.

0021–8928/92 $24.00+.00

ON THE STABILITY OF STEADY MOTIONS OF SYSTEMS WITH CYCLIC COORDINATES†

A. YA. KRASINSKII

Tashkent

(*Received* 14 *November* 1991)

A method of solving stabilization problems by isolating a controlled subsystem of possibly smaller dimension [1, 2] is developed further. The stabilizing action is determined by the solution of an optimal stabilization problem [3] for a linear controlled subsystem. The control that is found is implemented in the form of a feedback loop that uses an estimate [4] of the state vector (or part of it) constructed by measuring the perturbations of the positional coordinates. The stability of the unperturbed motion in a complete closed system is established by reducing the problem to a special case of the theory of critical cases [5, 6] or to the problem of stability under constantly acting perturbations [6].

IT WAS suggested in [7] that the steady motion of a system with cyclic coordinates could be stabilized by applying controlling actions to these coordinates. In Hamiltonian variables sufficient conditions have been obtained [7, 8] for the problem of obtaining asymptotic stability with respect to the positional coordinates and their momenta to be solvable. A qualitative analysis has been performed [9, 10] for the problem of stabilization in Lagrangian coordinates. Sufficient conditions have been formulated [11] for the asymptotic stabilization of steady motions. A range of criteria has been obtained [12] for controllability and observability for stabilization problems in Lagrangian variables. In the cases investigated [7–12] the controls have been applied over all cyclic coordinates, and asymptotic stabilization problems in the first approximation with respect to all phase variables have been considered [9–12].

Stabilization problems are investigated below which contain the weaker requirement of only stability of unperturbed motion. The control acts only on some of the cyclic coordinates.

1. Consider a mechanical system constrained by time-dependent geometrical constraints, and whose position is given by generalized coordinates $q_1, \ldots, q_n$. Here the kinetic energy of the system has the form (assuming that T does not depend explicitly on time)

$$T = T_2 + T_1 + T_0, \quad T_2 = \tfrac{1}{2} a_{ij}(q) \dot{q}_i \dot{q}_j$$
$$T_1 = d_j(q) \dot{q}_j, \quad T_0 = T_0(q), \quad q' = (q_1, \ldots, q_n)$$

The prime denotes transposition; summation is performed over repeated indices; the indices vary as follows:

$$i, j = 1, \ldots, n, \quad \rho, \nu = 1, \ldots, k, \quad r, s = k+1, \ldots, n$$
$$u, v = k+1, \ldots, k+m, \quad \omega, \epsilon = k+m+1, \ldots, n$$

Suppose the system is acted on by potential forces with energy $\Pi(q)$ and non-potential generalized forces $Q_i(q, q^{\bullet})$. We will assume that in some open domain of phase space $a_{ij}(q)$, $d_j(q)$, $T_0(q)$, the potential energy $\Pi(q)$ and the non-potential generalized forces $Q_i(q, q^{\bullet})$ are analytical functions of their variables, and that T_2 is a positive-definite function of the velocities.

† *Prikl. Mat. Mekh.* Vol. 56, No. 6, pp. 939–950, 1992.

We introduce the vectors and matrices

$$\alpha' = (q_1, \ldots, q_k), \quad \beta' = (q_{k+1}, \ldots, q_n), \quad \alpha_1' = (\dot{q}_1, \ldots, \dot{q}_k)$$
$$a_1(q) = \| a_{\rho\nu}(q) \|, \quad a_{12}(q) = \| a_{\rho r}(q) \| = a_{21}'(q), \quad a_2(q) = \| a_{rs}(q) \|$$
$$d_\alpha'(q) = \| d_1(q), \ldots, d_k(q) \|, \quad d_\beta'(q) = \| d_{k+1}(q), \ldots, d_n(q) \|$$

[In actual cases, decomposition of the vector q into vectors α and β is performed according to the various ways the kinetic energy, potential energy and generalized forces $Q_i(q, q^\cdot)$ depend on the generalized coordinates.]

We take as variables describing the state of the system the Routhian variables α, α_1, β and $p = \partial T / \partial \beta^\cdot = a_{21}\alpha_1 + a_2\beta^\cdot + d_\beta$. We introduce the Routhian function and write the equations of motion

$$R = T - \Pi - p'\beta^\cdot = \tfrac{1}{2}\alpha_1' a^* \alpha_1 + (d_\alpha' - d_\beta'\gamma' + p'\gamma')\alpha_1 -$$
$$- \tfrac{1}{2}(p' - d_\beta') b (p - d_\beta) + T_0 - \Pi(q), \qquad b(q) = a_2^{-1}(q)$$
$$a^*(q) = a_1 - \gamma a_{21}, \quad \gamma(q) = a_{12} b$$
$$\alpha^\cdot = \alpha_1, \quad a^*\alpha_1^\cdot = -\alpha_1'(a^*_{(\alpha)} - a^*_{(\beta)}\gamma - \tfrac{1}{2}a^*_{[\alpha]})\alpha_1 -$$
$$- \alpha_1'[a^*_{(\beta)} b(p - d_\beta) + \frac{\partial d_\alpha}{\partial \alpha} - \frac{\partial d_\alpha}{\partial \beta}\gamma' + \gamma \frac{\partial d_\beta}{\partial \beta}\gamma' - \gamma' \frac{\partial d_\beta}{\partial \alpha} +$$
$$+ (\gamma'_{(\alpha)})'(p - d_\beta) - \gamma\gamma'_{(\beta)}(p - d_\beta) - d_{\alpha[\alpha]} + \gamma d_{\beta[\alpha]} - \gamma_{[\alpha]}(p - d_\beta)] -$$
$$- [\frac{\partial d_\alpha}{\partial \beta} b - \gamma \frac{\partial d_\beta}{\partial \beta} b - d_\beta'(\gamma'_{(\beta)} b + b(\gamma'_{(\beta)})') - (d_{\beta[\alpha]})'b - d_\beta' b_{[\alpha]}] p -$$
$$- \gamma p - p'(\gamma_{(\beta)} b + \tfrac{1}{2} b_{[\alpha]}) p + \frac{\partial d_\alpha}{\partial \beta} b\, d_\beta - \gamma \frac{\partial d_\beta}{\partial \beta} b\, d_\beta -$$
$$- d_\beta' \gamma'_{(\beta)} b\, d_\beta - (d_{\beta[\alpha]})' b\, d_\beta - \frac{1}{2} d_\beta' b_{[\alpha]} d_\beta + \frac{\partial T_0}{\partial \alpha} - \Pi_\alpha + Q_\alpha(q, \alpha_1, p)$$
$$\beta^\cdot = -\gamma'\alpha_1 + b(p - d_\beta)$$
$$p^\cdot = \tfrac{1}{2}\alpha_1' a^*_{[\beta]}\alpha_1 + \alpha_1'(d_{\alpha[\beta]} - \gamma d_{\beta[\beta]} + \gamma_{[\beta]}(p - d_\beta) +$$
$$+ (d'_{\beta[\beta]} b + d_\beta' b_{[\beta]}) p - \tfrac{1}{2} p' b_{[\beta]} p - \tfrac{1}{2} d_\beta' b_{[\beta]} d_\beta -$$
$$- d'_{\beta[\beta]} b\, d_\beta + \frac{\partial T_0}{\partial \beta} - \Pi_\beta + Q_\beta(q, \alpha_1, p)$$
$$\Pi_\alpha = \partial \Pi / \partial \alpha, \quad \Pi_\beta = \partial \Pi / \partial \beta$$
$$Q_\alpha'(q, \alpha_1, p) = (Q_1, \ldots, Q_k), \quad Q_\beta'(q, \alpha_1, p) = (Q_{k+1}, \ldots, Q_n)$$

where, for any matrix $X(q) = \|x_{ij}(q)\|$, $X_{(q)}$ and $X_{[q]}$ denote "vectors" with matrix components $\|\partial x_{i\nu}/\partial q_j\|$ and $\|\partial x_{ij}/\partial q_\nu\|$, respectively, where ν is the number of the component of the "vector", and $X'_{(\alpha)}$ and $(X'_{(\alpha)})'$ are vectors with components $\|\partial x_{\nu i}/\partial q_j\|$ and $(\|\partial x_{\nu i}/\partial q_j\|)'$, respectively.

2. We will assume that the coordinates $\beta' = (q_{k+1}, \ldots, q_n)$ are cyclic, i.e. the kinetic energy, force function and generalized forces do not depend on them, and there are no generalized forces corresponding to the coordinates β. The system then has cyclic integrals and under given conditions can perform steady motions

$$q_\rho = q_\rho^\circ = \text{const}, \quad p_r = \delta_r = \text{const} \tag{2.1}$$

We know that motion (2.1) is always stable with respect to perturbations of the cyclic momenta (without the application of supplementary controls). Keeping this property of the natural proper motions of the system in mind, we consider the problem of stabilizing an unstable motion of (2.1) to one that is stable with respect to all phase variables by applying linear controls to some of the cyclic coordinates. We shall construct controls of possibly smaller dimension and a structure such that a smaller amount of information may be required. We introduce the notation $q_\rho = q_\rho^\circ + x_\rho$, $p_u = \delta_u + y_u$, $p_\omega = \delta_\omega + z_\omega$ for the perturbations and compose the equations for the perturbed motion, separating out the first approximation

$$\begin{aligned} &x^{\cdot} = x_1, \quad Ax_1^{\cdot} + (G+D)x_1 + (C+P)x + \Gamma_1 y^{\cdot} + (H_1+B_1)y + \\ &+\Gamma_2 z^{\cdot} + (H_2+B_2)z = N(x, x_1, y, z, u), \quad y^{\cdot} = u, \quad z^{\cdot} = 0 \end{aligned} \tag{2.2}$$

$$A = \| \{a^*_{\rho\nu}\}_0 \|, \quad W(\alpha, p) = \tfrac{1}{2}(p' - d'_\beta)\, b\,(p - d_\beta) - T_0 + \Pi$$

$$C = \| c_{\nu\rho} \| = \left\| \left\{ \frac{\partial^2 W}{\partial q_\nu \partial q_\rho} \right\}_0 \right\|, \quad H_1 = \left\| \left\{ \frac{\partial^2 W}{\partial q_\nu \partial p_u} \right\}_0 \right\| =$$

$$= \left\| \left\{ \frac{\partial}{\partial q_\nu} [(\delta_v - d_v) b_{uv} + (\delta_\omega + d_\omega) b_{u\omega}] \right\}_0 \right\|, \quad H_2 = \left\| \left\{ \frac{\partial^2 W}{\partial q_\nu \partial p_\epsilon} \right\}_0 \right\| =$$

$$= \left\| \left\{ \frac{\partial}{\partial q_\nu} [(\delta_v - d_v) b_{v\epsilon} + (\delta_\omega - d_\omega) b_{\omega\epsilon}] \right\} \right\|, \quad G = \| \{k_{\nu\rho}\}_0 \|$$

$$K(\alpha, p) = \| k_{\nu\rho} \| = \left\| \frac{\partial d_\nu}{\partial q_\rho} - \gamma_{\nu r} \frac{\partial d_r}{\partial q_\rho} + \frac{\partial \gamma_{\nu r}}{\partial q_\rho}(p_r - d_r) - \frac{\partial d_\rho}{\partial q_\nu} + \gamma_{\rho r} \frac{\partial d_r}{\partial q_\nu} - \right.$$

$$\left. - \frac{\partial \gamma_{\rho r}}{\partial q_\nu}(p_r - d_r) \right\|, \quad \Gamma_1 = \| \{ a_{\rho u} b_{uv} + a_{\rho\omega} b_{\omega v} \}_0 \|, \quad B_1 = \left\| \left\{ \frac{\partial Q_\alpha}{\partial p_u} \right\}_0 \right\|$$

$$\Gamma_2 = \| \{ a_{\rho u} b_{u\omega} + a_{\rho\omega} b_{\omega\epsilon} \}_0 \|, \quad P = \left\| \left\{ \frac{\partial Q_\alpha}{\partial \alpha} \right\}_0 \right\|, \quad D = \left\| \left\{ \frac{\partial Q_\alpha}{\partial \alpha_1} \right\}_0 \right\|$$

$$B_2 = \left\| \left\{ \frac{\partial Q_\alpha}{\partial p_\omega} \right\}_0 \right\|, \quad K'^1 = K(\alpha, p) - G, \quad A_* = a^*(q) - A$$

$$Q_\alpha(\alpha, \alpha_1, p) = \{Q_\alpha\}_0 + Px + Dx_1 + B_1 y + B_2 z + Q'^2_\alpha$$

$$\Gamma'^1 = \| a_{\rho u} b_{uv} + a_{\rho\omega} b_{\omega v} \| - \Gamma_1, \quad \frac{\partial W}{\partial \alpha} = \left\{ \frac{\partial W}{\partial \alpha} \right\}_0 + Cx + H_1 y + H_2 z + \Pi'^2$$

$$\Phi(x, x_1, y, z, u) = -x_1'(a^*_{(\alpha)} - \tfrac{1}{2} a^*_{[\alpha]}) x_1 - \Pi'^2 - \Gamma'^1 u - K'^1 x_1 + Q'^2_\alpha$$

$$L = -(G+D)x_1 - (C+P)x - (H_1+B_1)y - (H_2+B_2)z - \Gamma_1 u$$

$$N(x, x_1, y, z, u) = \Phi + AA_*^{-1}(L + \Phi)$$

The notation $\{\ldots\}_0$ means that the expression is calculated for the motion (2.1); a superscript after the prime gives the order of the subsequent terms in the expansion of the corresponding expression. It is obvious that in order for motion (2.1) to exist it is necessary that

$$\left\{ \frac{\partial W}{\partial \alpha} \right\}_0 + \left\{ Q_{\alpha\,0} \right\} = 0$$

be satisfied.

Remark 2.1. The matrix $C+P$ of coefficients of the linear positional forces, unlike the cases considered previously [1, 2, 11, 12], is not in general symmetric. A skew-symmetric component does not only appear in the presence of non-potential positional forces in the vector Q_α, but can also appear under the action of forces containing cyclic velocities. Suppose, for example, that only forces linear with respect to the velocities are present

$$Q_\alpha(\alpha, \alpha^{\cdot}, p) = D_1(\alpha)\alpha_1 + D_2(\alpha)\beta = (D_1 - D_2\gamma')\alpha_1 + D_2 b(p - d_\beta)$$

Separating out the linear terms in the expansion of this vector in the neighbourhood of motion (2.1), we obtain

$$Q_\alpha = \{Q_\alpha\}_0 + [\{\frac{\partial}{\partial\alpha}\{D_2 b\}\}_0(\delta - \{d_\beta\}_0) - \{D_2 b\}_0 \{\frac{\partial d_\beta}{\partial\alpha}\}_0]x +$$
$$+ \{D_1 - D_2\gamma'\}_0 x_1 + \{D_2 b\}_0 y_1 + Q_\alpha'^2, \qquad y_1' = (y', z')$$

and we see that the appearance of a skew-symmetric component is even possible from the action of forces that are linear in the cyclic velocities with constant coefficients, if the kinetic energy depends on the coordinates.

Isolating the controlled subsystem

$$\xi^\cdot = F\xi + \Psi u, \qquad \xi' = (x', x_1', y'), \qquad \Psi' = (0 - \Gamma_1' A^{-1}, E_m) \tag{2.3}$$

$$F = \begin{Vmatrix} F_1 & 0 \\ & -A^{-1}(H_1 + B_1) \\ 0 & 0 \end{Vmatrix}, \qquad F_1 = \begin{Vmatrix} 0 & 0 \\ -A^{-1}(C+P) & -A^{-1}(G+D) \end{Vmatrix}$$

we obtain a criterion for the stabilizability [3] of motion (2.1)

$$\operatorname{rank} Y = \operatorname{rank}(\Psi F \Psi \ldots F^{2k+m-1}\Psi) = 2k + m$$

which, using the structure of the final row of the coagulated matrix Y, can be written in the form

$$\operatorname{rank} Y_1 = \operatorname{rank}(\Psi_1 F_1 \Psi_1 \ldots F_1^{2k-1}\Psi_1) = 2k, \tag{2.4}$$
$$\Psi_1' = (-\Gamma_1' A^{-1}, \ [\Gamma_1' A^{-1}(G+D)' - H_1 - B_1']A^{-1})$$

We introduce the quality criterion

$$I = \int_0^\infty [\Omega_1(\xi) + \Omega_2(u)]\, dt \tag{2.5}$$

(Ω_1 and Ω_2 being positive definite quadratic forms of their arguments).

Theorem 2.1. Condition (2.4) is sufficient for the stabilizability of motion (2.1) by linear controls applied to some of the cyclic coordinates under the action of positional cordinate potential forces with energy $\Pi(q)$ and arbitrary non-potential forces $Q_\alpha(\alpha, \alpha_1, p)$. The stabilizing action $u^* = M\xi$ can be found by solving the problem of optimizing [in the sense of minimizing integral (2.5)] the stabilization problem for the controlled subsystem (2.3) and depends only on variables occurring in this subsystem.

The proof is similar to the proof of the theorem in [2].

In the full system (2.2) under the action of u^* there is a critical case whose reduction in general requires [5, 6] a non-linear transformation of the variables x, y. As a result of this the control u^* ensures asymptotic stability of the positional velocities, and, generally speaking, the stability of the positional coordinates and cyclic velocities.

Remark 2.2. If the control is applied to the whole cyclic coordinate vector, condition (2.4) becomes the condition for the asymptotic stabilizability of motion (2.1) under action on positional coordinates and with arbitrary non-potential generalized forces as well as potential forces.

Remark 2.3. From the structure of the coagulated matrix Y_1 one can see that the controllability depends very much on the coefficients of the linear terms in the expansion of the generalized non-potential forces. Here some new cases can appear in addition to those considered earlier [2, 12]. For example, for gyroscopically decoupled systems (in the chosen part of the cyclic coordinates whose momentum perturbations occur in the controlled subsystem) ($\Gamma_1 = 0$) the possibility arises of stabilizing the trivial motions ($H_1 = 0$) in these coordinates for $B_1 \neq 0$. In particular, we have the following corollary.

Corollary 2.1. If for $H_1 = 0$ the rank of matrix B_1 equals the number of positional coordinates, a gyroscopically decoupled subsystem (with respect to the controlled cyclic coordinates) is always controllable (see Theorem 2.1 of [12]).

In motion-stabilization problems for systems with several cyclic coordinates the question arises of reducing the dimension of the controlling action. Using the structure of the equations of perturbed motion in [13], we shall estimate the number of controlled cyclic coordinates.

Corollary 2.2. The smallest number of controlled cyclic coordinates required to satisfy the sufficiency condition (2.4) for the stabilizability of motion (2.1) is equal to the number r of non-trivial polynomials of the matrix F_1.

Remark 2.4. The problem being considered is considerably different from that investigated in [14] for the smallest dimension of a control vector stabilizing the null solution of the non-linear system (with an isolated linear part)

$$x = f(x) + \varphi(x, u) = Ax + Bu + \dots$$

up to asymptotic stability and in the first approximation with respect to all variables. In the problem under consideration, when there is a change in the control dimension, the dimension of the whole controlled subsystem changes, as a result of which there is a change in both matrices F and Ψ that play the role of matrices A and B. (In [14], when the dimension of u changes, only the number of columns in matrix B changes.) Furthermore, the matrix F here has at least m zero eigenvalues.

Thus when solving specific stabilization problems the verification that criterion (2.4) is satisfied should begin with $m = r$. When there are several cyclic momenta, to each selection of controlled momenta there corresponds a matrix Ψ_1, because its second row contains the matrix $H_1 + B_1$ and, moreover, the matrix Γ_1 will also change. If however the total number of cyclic coordinates is less than r, the sufficiency condition (2.4) cannot be satisfied.

3. In many papers on stabilization it is implicitly assumed that at each instant of time all variables that are necessary to construct the control are known. However, it is difficult to believe that in the majority of practical situations all the required components of the state vector are accessible to measurement (either because there is a limited number of measurement devices, or else some of the state variables are in principle impossible to measure—for example, y and z in the general case). The output from the controlled object usually consists of individual state vector components or linear combinations of such components. Hence, in order to make use of the possibilities supplied by control with feedback from the state, it is necessary to find an acceptable estimate [4] for the whole state vector (or some of it) from the output data. The problem of stabilization to asymptotic stability in Lagrangian coordinates of steady motions in all the phase variables of the problem was investigated from this point of view in [12]. The controls depended on all phase variables and acted on all cyclic coordinates.

In the stabilization problems considered here, the controls are only applied to some of the cyclic coordinates and depend only on the phase variables of the controlled subsystem, which need not contain perturbations of many cyclic momenta. It is natural also to expect a reduction in the amount of measured information necessary to produce the stabilizing actions in such a method of stabilization. Here one should pay attention to the qualitative difference in the meaning of direct observability (measurement) of the variables y, z (perturbations of cyclic momenta) from variables x, x_1, because to obtain the values of just one of the components y, z may require information on all the positional coordinates of α and all (including cyclic) velocities $\alpha^{\bullet}$, $\beta^{\bullet}$.

Hence we consider therefore the observability problem in system (2.2) without measuring perturbations of the cyclic momenta y, z. To be specific we will investigate the problem of the possibility of constructing controls when only measurements of information solely on perturbations of positional coordinates are available.

Assertion 3.1. If the condition

$$\operatorname{rank}(H_1 + B_1, H_2 + B_2) = n - k \tag{3.1}$$

is satisfied, system (2.2) is completely observable in a neighbourhood of motion (2.1) in terms of measurements of perturbations of positional coordinates.

Assertion 3.2. If in the equations of perturbed motion (2.2) there are no terms linear in uncontrolled momentum perturbations (i.e. $H_2 + B_2 = 0$), then with the condition

$$\operatorname{rank}(H_1 + B_1) = m \tag{3.2}$$

the variables x, x_1, y in system (2.2) are observable in a neighbourhood of the unperturbed motion (2.1) through measurements of positional coordinate perturbations.

The validity of the assertion follows from the structure of system (2.2) and [15].

Remark 3.1. Condition (3.1) for the complete observability of the system cannot be satisfied if the number of positional coordinates is less than the number of cyclic coordinates: the rank of a matrix cannot be greater than the number of its rows. For $Q_\alpha = D_1\alpha_1$, $T = T_2$, a similar result in Lagrangian coordinates was obtained in [12].

Remark 3.2. If system (2.2) contains no terms linear in z, then, like the preceding, to satisfy condition (3.2) it is necessary that the number of positional coordinates is not less than the number of controlled cyclic momenta, i.e. $k \geqslant m$.

Remark 3.3. According to conditions (3.1) and (3.2) the non-potential generalized forces also influence the observability. In particular, unlike in [12], complete observability of system (2.2) in terms of measurements of positional coordinate perturbations is possible in a neighbourhood of the trivial ($H_1 = 0$, $H_2 = 0$) motion. For this it is sufficient that rank $(B_1, B_2) = n - k$.

Under condition (3.1) for the system

$$\eta^{\cdot} = F_2\eta + \Psi_2 u, \quad \sigma = S_1\eta, \quad \Psi_2' = (\Psi', 0), \quad S = (E_k, 0, 0) \tag{3.3}$$

$$\eta' = (\xi', z'), \quad S_1 = (S, 0), \quad F_2 = \begin{Vmatrix} F & H_2 + B_2 \\ 0 & 0 \end{Vmatrix}$$

there exists an asymptotic identifier (a system of asymptotic estimates) [14]

$$\eta^{\cdot\circ} = F_2\eta^\circ + L(\sigma - S_1\eta^\circ) + \Psi_2 u$$

of the state vector η in terms of the measurement σ.

For $H_2 + B_2 = 0$ and condition (3.2), there exists for the controlled subsystem (2.3) an asymptotic identifier

$$\xi^{\cdot\circ} = F\xi^\circ + L_1(\sigma_1 - S\xi^\circ) + \Psi u \tag{3.4}$$

of the state vector of this subsystem in terms of the measurement $\sigma_1 = S\xi$.

Here the constant matrices L and L_1 of appropriate dimensions, determining the form of the approach of the estimation errors to zero, can be found by solving the optimal stabilization problem for the systems

$$\zeta^{\cdot} = F_2'\zeta + S_1' w \tag{3.5}$$

$$\mu^{\cdot} = F'\mu + S'v \tag{3.6}$$

respectively, for specified quadratic quality criteria, which follows from the duality of the control and observation problems [4] for systems (3.3) and (3.5) (corresponding to (2.3) and (3.6) in [3]).

Remark 3.4. This paper uses, for simplicity, just one sufficiently rich set of external information: measurements of perturbations of all positional coordinates, which is, of course, not necessary in general. In particular, one can measure perturbations of only some of the positional coordinates. Then one should take as the matrix S in systems (3.3) and (3.4) the matrix

$$S_* = (I_*, 0, 0), \quad I_* = \begin{Vmatrix} E_l & 0 \\ 0, & 0 \end{Vmatrix}, \quad l < k$$

and instead of conditions (3.1) and (3.2) one should, respectively, require that the following conditions should be satisfied

$$\operatorname{rank}(S_{1*}', F_2' S_{1*}' \dots F_2'^{\,n+k-l} S_{1*}') = n + k, \quad S_{1*} = (S_*, 0)$$

$$\operatorname{rank}(S_*' F' S_*' \dots F'^{\,2k+m-l} S_*') = 2k + m$$

Using specific properties of system (2.2) of the equations of perturbed motion, one can reduce the dimension of the estimation system, taking (3.4) to be the identifier of the system irrespective of the presence of terms linear in z in Eqs (2.2).

Theorem 3.1. Suppose that for a mechanical system described by Eqs (2.2), conditions (2.4) and (3.2) are satisfied in a neighbourhood of the unperturbed motion (2.1). Then motion (2.1) can be stabilized with respect to all variables by the application of the linear control $u = M\xi^\circ$ to some of the cyclic variables, where the matrix M is given by the solution of the optimal stabilization problem for the controlled subsystem (2.3) and ξ° is the estimate of the vector ξ obtained by the estimation system (3.4) from the measurement σ_1. The matrix L_1 is found by solving the optimal stabilization problem for system (3.6).

Proof. Under condition (2.4) for the system

$$\xi^{\cdot} = F\xi + \Psi u + N_2, \quad N_2 = N_1 \mid_{z=0}, \quad N_1' = (0, N'A^{-1}, 0) \tag{3.7}$$

the control $u^* = M\xi$, given by the solution of the optimal stabilization problem for subsystem (2.3) according to the criterion (2.5), supplies asymptotic stability for the solution $\xi = 0$ in the first approximation. From (15) and the structure of the system, condition (3.2) is a sufficient condition for the observability of system (3.7) in a neighbourhood of the motion $\xi = 0$ from the measurement σ_1. The identifier (3.4) then exists [4], where the matrix L_1 can be determined [3] from the solution of the optimal stabilization problem for the solution $\mu = 0$ of system (3.6) with the criterion

$$I_1 = \int_0^\infty [\Omega_3(\mu) + \Omega_4(\upsilon)]\,dt$$

where Ω_3 and Ω_4 are positive definite quadratic forms. Hence, in the closed system

$$\begin{aligned} &\xi^{\cdot} = F\xi + \Psi u + N_2, \quad \sigma_1 = S\xi \\ &\xi^{\cdot\circ} = F\xi^\circ + L_1(\sigma_1 - S\xi^\circ) + \Psi u, \quad u = M\xi^\circ \end{aligned} \tag{3.8}$$

the real parts of all roots of the characteristic equation are negative [4, Theorem 7.7]. Consequently, the solution $\xi = 0$, $\xi^\circ = 0$ of system (3.8) is asymptotically stable. The closed system

$$\begin{aligned} &\xi^{\cdot} = F\xi + \Psi u + Hz + N_1, \quad z^{\cdot} = 0, \quad \sigma_1 = S\xi, \quad u = M\xi^\circ \\ &\xi^{\cdot\circ} = F\xi^\circ + L_1(\sigma_1 - S\xi^\circ) + \Psi u, \quad H' = (0, -(H_2 + B_2)'A^{-1}, 0) \end{aligned} \tag{3.9}$$

of the complete problem is obtained from system (3.8) by the action of constant perturbations occurring at $z(t) = z_0 = \text{const} \neq 0$. According to the theorem on stability under constantly acting perturbations [6, Sec. 70] the point $\xi = 0$, $\xi^\circ = 0$ for system (3.9) is stable.

Remark 3.5. We note the connection between the result obtained and Routh's theorem [16] on the conditional stability of steady motions and Lyapunov's [17] supplement to this theorem. Equation (3.7) describes the (controlled) perturbed motion of systems in a neighbourhood of motion (2.1) with unperturbability of uncontrolled cyclic momenta. Hence the closed system (3.8) provides asymptotic stability for motion (2.1) to first order of approximation under the condition $z_0 = 0$ on the initial perturbations. When this condition is removed the motion becomes (unconditionally) non-asymptotically stable.

Remark 3.6. According to the theorem that has been proved, conditions (2.4) and (3.2) are sufficient conditions for the problem of controlled stability with feedback from estimates of the state vector to be solvable irrespective of whether system (2.2) contains terms linear in the perturbations of the uncontrolled momenta (cf. with Assertions 3.1 and 3.2). When these conditions are satisfied not only is the dimension of the control problem reduced compared with [7–12], but the dimension of the identifier is reduced compared with [12]. The total dimension of the linear closed system for which the matrices M and L_1 are determined, is equal to $2(2k + m)$ as opposed to $2(n + k)$ in [12], i.e. a reduction of twice the number of uncontrolled cyclic momenta. But in the problem under consideration, unlike the one studied in [12], there will only be non-asymptotic stability in the closed system.

Remark 3.7. For the unique determination of the matrices L and L_1 from the solution of the dual stabilization problem the methods of synthesizing stabilization laws considered in [18] may turn out to be efficient (see Example 5.1 below).

4. We will investigate the possibility of stabilizing motion (2.1) by using linear controls that are independent of cyclic momentum perturbations to some of the cyclic variables

$$u_1 = M_1 x + M_2 x_1 \tag{4.1}$$

where the matrices M_1 and M_2 are to be determined. In the controlled subsystem

$$x^{\cdot} = x_1, \quad x_1^{\cdot} = -A^{-1}(C+P)x - A^{-1}(G+D)x_1 - A^{-1}\Gamma_1 u_1 - A^{-1}(H_1+B_1)_y \tag{4.2}$$

two fundamentally different situations are possible. The first of these is characterized by the fact that an action (4.1) exists asymptotically stabilizing the point

$$x = 0, \quad x_1 = 0, \quad y = 0 \tag{4.3}$$

by virtue of Eqs (4.2) for all the variables of this system. One can clarify whether such a possibility exists using the Routh–Hurwitz criterion for the equation

$$\begin{vmatrix} E_k \lambda & -E_k & 0 \\ C+P+\Gamma_1 M_1 & A\lambda + D + G + \Gamma_1 M_2 & H_1 + B_1 \\ -M_1 & -M_2 & E_m \lambda \end{vmatrix} = 0 \tag{4.4}$$

Obviously, for this to be true when $\det(C+P) \neq 0$ it is necessary that

$$\det(C+P)\det(M_1(C+P)^{-1}(H_1+B_1)) > 0 \tag{4.5}$$

because this determinant is equal to the free term in Eq. (4.4).

Remark 4.1. For $m>k$ the determinant (4.5) vanishes [19]. Hence a control of the form (4.1) cannot ensure asymptotic stabilizability if the number of cyclic coordinates is greater than the number of positional coordinates.

Remark 4.2. By virtue of (4.2) there are in general no efficient algorithms for constructing controls of the form (4.1) that ensure asymptotic stabilizability of the point (4.3). The coefficients of controlled phase variables can be made to vanish by choosing the coefficients in the quality criterion [20]. To determine the matrices M_1 and M_2 uniquely one can use the suboptimal control construction method [21]. But it is impossible to guarantee that such a control can be found in the general case, because the satisfaction of the Routh–Hurwitz criterion for Eq. (4.4) is only a necessary condition for the existence of suboptimal control.

We will now analyse the second situation possible in system (4.2) under the action of control (4.1), which is characterized by the fact that such controls cannot, in principle, ensure asymptotic stability of the point (4.3) because of system (4.2). One of the simplest cases of this kind appears when $H_1 + B_1 = 0$. For systems that are gyroscopically coupled with respect to the controlled momenta (i.e. $\Gamma_1 \neq 0$), under special conditions it is still possible to stabilize motion (2.1) by control (4.1) if we take the following as the controlled subsystem

$$\xi_1^{\cdot} = F_1 \xi + Q_1 u_1, \quad \xi_1' = (x', x_1'), \quad Q_1' = (0, -\Gamma_1' A^{-1}) \tag{4.6}$$

Theorem 4.1. If, for a mechanical system that is gyroscopically coupled in some of its cyclic momenta, with the condition

$$\operatorname{rank}(Q_1 F_1 Q_1 \dots F_1^{2k-1} Q_1) = 2k \tag{4.7}$$

the equation of perturbed motion (2.2) does not contain freely-entering perturbations of the cyclic momenta, then the unperturbed motion (2.1) is asymptotically stabilized in perturbations of the positional coordinates and their velocities and stabilized in the cyclic velocities under the application to some of the cyclic coordinates of the linear control $u_2 = M_3 \xi_1^{\circ}$. The control is formed through feedback via the estimate ξ_1° obtained for the vector ξ_1 from the measurement $\sigma_2 = S_2\xi_1$, $S_2 = (E_k, 0)$ from the system

$$\xi_1^{\cdot\circ} = F_1\xi_1^\circ + L_2(\sigma_2 - S_2\xi_1^\circ) + Q_1 u$$

where the matrix L_2 is determined by the solution of the stabilization problem for the system

$$\dot{\nu} = F_1'\nu + S_2'w_1 \tag{4.8}$$

Proof. The asymptotic stability of the closed system

$$\xi_1^{\cdot} = F_1\xi_1 + Q_1u_2 + N_4, \quad \sigma_2 = S_2\xi_1, \quad N_4 = N_3|_{z,y=0}, \quad N_3' = (0\,N'A^{-1})$$
$$\xi_1^{\cdot\circ} = F_1\xi_1^\circ + L_2(\sigma_2 - S_2\xi_1^\circ) + Q_1u_2, \quad u_2 = M_3\xi_1^\circ$$

is proved as in Theorem 3.1, because with condition (4.6) the problem of the optimal stabilization of the point $\xi_1 = 0$ is solvable [3] for the given criterion

$$\int_0^\infty [\Omega_5(\xi_1) + \Omega_6(u_1)]\,dt$$

[$\Omega_5(\xi_1)$ and $\Omega_6(u_1)$ are positive definite quadratic forms.] Observability from the measurement σ_2, as can easily be verified, occurs for arbitrary matrices in the second row of the coagulated matrix F_1. Consequently, one can determine [3, 18] the matrix L_2 by solving the problem of stabilizing system (4.8). The stability specified in the statement of the theorem in the closed system

$$\begin{aligned} &\xi_1^{\cdot} = F_1\xi_1 + Q_1u_2 + N_3, \quad \sigma_2 = S_2\xi_1, \quad y_1^{\cdot} = u_2, \quad z^{\cdot} = 0 \\ &\xi_1^{\cdot\circ} = F_1\xi_1^\circ + L_2(\sigma_2 - S_2\xi_1^\circ) + Q_1u_2, \quad u_2 = M_3\xi_1^\circ \end{aligned} \tag{4.9}$$

follows from the Lyapunov–Malkin theorem [5, 6] on stability in the special case of $n-k$ zero roots.

Remark 4.3. If y and z occur in Eqs (2.2) free from x and x_1 only in the non-linear terms, then linear control of u_2 cannot ensure stabilization of motion (2.1) when $m = 1$. Under the action of u_2 and the presence of terms non-linear in y and z the problem of the stability of the null solution of system (4.9) reduces to the problem of the stability of the null solution of the system

$$\xi_1^{\cdot} = F_1\xi_1 + Q_1u_2, \quad y^{\cdot} = u_2 \tag{4.10}$$

for constantly acting perturbations of $z = z_0 = \text{const}$.

For stability in system (4.9) one must have [6] asymptotic stability of the null solution of system (4.10) at $z_0 = 0$. Such stability is possible for $m = 1$ when [5, 6] y occurs in the non-linear terms free from x and x_1 to odd degree. Here however this degree is only equal to two.

5. *Example 5.1.* As an application of the stabilization method presented in Theorem 4.1, we will solve in general form the problem of stabilizing steady motions with one positional and several cyclic coordinates in a situation included in this theorem. Thus, we have the equations of perturbed motion

$$x^{\cdot} = x_1, \quad x_1^{\cdot} = ax + bx_1 + gu + N_5(x, x_1, y, z, u), \quad y^{\cdot} = u, \quad z^{\cdot} = 0 \tag{5.1}$$

where a, b and g are constants, N_5, x, x_1 and y are scalars, z is a vector, with $g \neq 0$, and $N_5(0,0,y,z,0) \equiv 0$. Introducing the simplest quality criterion

$$I = \int_0^\infty (x^2 + x_1^2 + u^2)\,dt \tag{5.2}$$

we find (see Eqs (111.13) of [3]) the coefficients of the Lyapunov functional that are optimal in the sense of minimizing (5.2), from which we obtain

$$u_2 = g^{-1}[dx^0 + (b + (b^2 + g^2 + 2d)^{1/2}x_1^0], \quad d = a + (a^2 + g^2)^{1/2} \tag{5.3}$$

Here the vector (x°, x_1°) is an estimate for the vector (x, x_1) obtained from the system

$$\begin{aligned} &x^{\cdot\circ} = l_1(x - x^\circ) + x_1^\circ, \quad l_1 = \alpha + b \\ &x_1^{\cdot\circ} = ax^\circ + bx_1^\circ + l_2(x - x^\circ) + gu, \quad l_2 = \tfrac{1}{2}(\alpha + b)(\alpha + 2b) \end{aligned} \tag{5.4}$$

(where α is a sufficiently large positive number) for measuring x. The matrix $L'_3 = (l_1, l_2)$ is found by solving the dual stabilization problem by the method proposed by Krasovskii (see [18, p. 97]).

Example 5.2. Consider an asymmetric gyroscope. The Routhian function has the form [2]

$$R = R_2 + R_1 + R_0 = \tfrac{1}{2}[a - \Delta^{-1}(b_{22}c_1^2 - 2b_{12}c_1c_2 + b_{11}c_2^2)]\,\theta'^2 + \Delta^{-1}[b_{22}c_1 - b_{12}c_2)p_1 +$$
$$+ (b_{11}c_2 - b_{12}c_1)p_2]\,\theta + (m_2 + m_3)gy_2 \sin\epsilon\cos\theta - \tfrac{1}{2}\Delta^{-1}[b_{22}p_1^2 - 2p_1p_2b_{12} + b_{11}p_2^2],$$
$$\Delta = b_{11}b_{22} - b_{12}^2$$

for gyroscope parameters in the notation used in [10]. The equations of the manifold of steady motions $\partial R_0/\partial\theta = 0$ have solutions other than the motion

$$\theta_0 = 0, \quad p_2 = \delta_2 = \text{const}, \quad p_1 = \delta_1 = \text{const} \tag{5.5}$$

In particular, with the additional condition $\lambda = 0$ (the axis of the flywheel parallel to the η_2 axis, it being previously [10] assumed that $\nu = 0$), for sufficiently large δ_2 a real solution

$$\begin{aligned}
&\theta_0 = \tfrac{1}{2}\pi, \quad p_2 = \delta_2, \quad p_1 = \tfrac{1}{2}\mu_1^{-1}(1 \pm \sqrt{1 - 4\mu_1(m_2 + m_3)gy_2\delta_2^{-2}})\,b_{11}^*\delta_2 \\
&b_{11}^* = J + A_2\cos^2\epsilon + B_0 + C_2\sin^2\epsilon + m_3(x_1^2\sin^2\epsilon + y_1^2 + z_1^2\cos^2\epsilon) \\
&\mu_1 = (G_2 + m_3x_1y_1)\cos\epsilon
\end{aligned} \tag{5.6}$$

exists. For $(G_2 + M_3x_1y_1)y_2\cos\epsilon > 0$ one can obtain from (5.6) the solution

$$\theta_0 = \tfrac{1}{2}\pi, \quad \tfrac{1}{2}\delta_2^* = \pm\sqrt{\mu_1(m_2 + m_3)gy_2}, \quad \delta_1^* = \tfrac{1}{2}\mu_1^{-1}b_{11}^*\delta_2^{*2} \tag{5.7}$$

The possibility of stabilizing motion (5.5) by applying controls to both cyclic coordinates was analysed in [10]. In [12] the measurements supplying information necessary for formulating controls stabilizing (5.5) were determined by the same method. For the system being considered, in Eqs (2.2) we have

$$\Gamma_1 = \{b_{22}c_1 - b_{12}c_2\}_0, \quad \Gamma_2 = \{b_{11}c_2 - b_{12}c_1\}_0, \quad H_1 + B_1 = \left\{\frac{\partial b_{12}}{\partial\theta}\delta_2\Delta^{-1} + \right.$$
$$\left. + b_{22}\frac{\partial\Delta}{\partial\theta}\delta_1\Delta^{-2} - b_{12}\frac{\partial\Delta}{\partial\theta}\delta_2\Delta^{-2}\right\}_0, \quad H_2 + B_2 = \left\{\frac{\partial b_{12}}{\partial\theta}\delta_1\Delta^{-1} - \frac{\partial b_{11}}{\partial\theta}\delta_2\Delta^{-1} + \right.$$
$$\left. + (b_{11}\delta_2 - b_{12}\delta_1)\Delta^{-2}\frac{\partial\Delta}{\partial\theta}\right\}_0$$

In a neighbourhood of motion (5.5) $\Gamma_1 \neq 0$, $\Gamma_2 \neq 0$, $H_1 + B_1 = 0$, $H_2 + B_2 = 0$, and the equations of perturbed motion do not contain freely occurring y and z. According to Theorem 4.1 the problem of stabilizing motion (5.5) is solved by applying the control given by (5.3) to one of the cyclic coordinates, where the vector (x°, x_1°) is obtained from system (5.4). We remark that for stabilizing the motion (5.5) with the help of the control constructed in [2], although the number of positional coordinates is equal to the number of cyclic coordinates occurring in the controlled subsystem, unlike the case in [2], condition (3.2) is not satisfied. Hence when stabilizing motion (5.5) by the control of [2], it is necessary, as in [12], to measure the perturbations of the cyclic velocities. In view of the non-satisfaction of condition (4.5) because $H_1 + B_1 = 0$ and $H_2 + B_2 = 0$, it is impossible to reduce the volume of information by excluding perturbations of the controlled momentum from the control.

In a neighbourhood of motion (5.7)

$$\Gamma_2 = 0, \quad \Gamma_1 = A_0[A_2 + B_0 + m_3(y_1^2 + z_1^2)]\cos\epsilon \neq 0, \quad H_2 + B_2 = [(m_2 + m_3)gy_2\,\mu^{-1}]^{1/2}\sin\epsilon$$

in which y and z occur freely. When the controllability condition

$$\begin{aligned}
&a_1g_\kappa^2 - bg_\kappa h_\kappa - h_\kappa^2 \neq 0, \quad \kappa = 1, 2, \quad g_\kappa = -A^{-1}\Gamma_\kappa, \quad h_\kappa = -A^{-1}(H_\kappa + B_\kappa) \\
&A = a - [b_{22}c_1^2(\theta_0) - 2b_{12}(\theta_0)c_1(\theta_0) - b_{11}(\theta_0)c_2]/\Delta(\theta_0), \quad a_1 = -A^{-1}\{\partial^2R_0/\partial\theta^2\}_0
\end{aligned}$$

is satisfied, the problem is solvable by using a linear moment along one of the cyclic coordinates. Stabilization by a control acting about the axis of the outer ring of the suspension requires measurement of the cyclic velocity perturbation because observability condition (3.2) is not satisfied. If however motion (5.7) is stabilized by a moment applied to the axis of proper rotation, the conditions of Theorem 3.1 are satisfied; to formulate the control it is sufficient just to measure x from which an identifier of the form (3.4) can be constructed.

In a neighbourhood of motion (5.6)

$$\Gamma_1 \neq 0, \quad \Gamma_2 = 0, \quad H_1 + B_1 = -(b_{11}^*)^{-1} \sin\epsilon\delta_2 \sqrt{1 - 4\delta_2^{-2}\mu_1(m_2 + m_3)gy_2} \neq 0$$
$$H_2 + B_2 = \delta_2 \sin\epsilon(1 \pm \sqrt{1 - 4\delta_2^{-2}(m_2 + m_3)\mu_1 gy_2}) \neq 0$$

The condition of complete observability (3.1) is not satisfied by virtue of Remark 3.1. Nevertheless, using Theorem 3.1 and satisfying (3.2), we can construct a closed system of the form (3.9) when measuring only x. In the problem of stabilizing motion (5.6) one can also construct a control of the form (4.1). Equation (4.4), stabilized by applying a moment about the proper axis of rotation, acquires the form

$$A\lambda^3 + d\lambda^2 + \lambda(m_2 H_3 + C) + m_1 H_3 = 0, \quad C = \{\partial^2 R_0/\partial\theta^2\}_0, \quad H_3 = H_2 + B_2$$

and from the Routh–Hurwitz criterion we have

$$m_2 H_3 + C > 0, \quad m_1 > (AH_3)^{-1} d(m_2 H_3 + C)$$

We note that unlike the preceding cases of stabilization, when the problem could be solved when there is no dissipation, i.e. when $d = 0$, in the last case dissipation with respect to the positional velocity is necessary.

The author thanks V. V. Rumyantsev for his interest and for helpful discussions, and also the referee for useful remarks.

REFERENCES

1. KRASINSKI A. Ya., On mathematical modelling in problems of optimal stabilization of steady motions. In *Proceedings of the All-Union Seminar School "Mathematical Modelling in Science and Technology"*, Perm: UNTs Akad. Nauk SSSR, pp. 184–185. 1986.
2. KRASINSKII A. Ya., On the stabilization of steady motions of mechanical systems with cyclic coordinates. *Prikl. Mat. Mekh.* **52,** 542–548, 1988.
3. KRASOVSKII N. N., Problems of the stabilization of controlled motions. In *Theory of Stability of Motion* (Edited by I. G. Malkin), pp. 475–514. Nauka, Moscow, 1966.
4. KALMAN R., FALB P. and ARBIB M., *An Outline of Mathematical Systems Theory.* Mir, Moscow, 1971.
5. LYAPUNOV A. M., *Collected Papers*, Vol. 2. Izd. Akad. Nauk SSSR, Moscow and Leningrad, 1956.
6. MALKIN I. G., *Theory of Stability of Motion.* Nauka, Moscow, 1966.
7. RUMYANTSEV V. V., On the control and stabilization of systems with cyclic coordinates. *Prikl. Mat, Mekh.* **36,** 966–976, 1972.
8. LILOV L. K., The stabilization of steady motions of systems with respect to some of the variables. *Prikl. Mat. Mekh.* **36,** 977–985, 1972.
9. SAMSONOV A. V., The stability of steady motions of a system with pseudocyclic coordinates. *Prikl. Mat. Mekh.* **45,** 512–520, 1981.
10. KLOKOV A. S. and SAMSONOV V. A., The stabilizability of trivial steady motions of gyroscopic coupled systems with pseudocyclic coordinates. *Prikl. Mat. Mekh.* **49,** 199–202, 1985.
11. ATANASOV V. A. and LILOV L. K., On the stabilizability of steady motions of a system with pseudocyclic coordinates. *Prikl. Mat. Mekh.* **52,** 713–718, 1988.
12. KALENOVA V. I., MOROZOV V. M. and SALMINA M. A., The problem of stabilizing steady motions of systems with cyclic coordinates. *Prikl. Mat. Mekh.* **53,** 707–714, 1989.
13. GABASOV R. and KIRILLOVA F. M., *Qualitative Theory of Optimal Processes.* Nauka, Moscow, 1971.
14. LILOV L. K., On determining the smallest number of controls that stabilize an equilibrium position. *Prikl. Mat. Mekh.* **34,** 788–795, 1970.
15. KRASOVSKII A. A., An observatory condition for non-linear processes. *Dokl. Akad Nauk SSSR.* **242,** 1265–1268, 1978.
16. ROUTH E. J., *Dynamics of a System of Rigid Bodies*, Part 2. Nauka, Moscow, 1983.
17. RUMYANTSEV V. V., *On the Stability of Stationary Satellite Motion.* VTs Akad. Nauk SSSR, Moscow, 1967.
18. FURASOV V. D., *Stability of Motion: Estimates and Stabilization.* Nauka, Moscow, 1977.
19. MISHINA A. P. and PROSKURYAKOV I. V., *Advanced Algebra.* Nauka, Moscow, 1965.
20. KRASOVSKII A. A., *Statistical Theory of Transitional Processes in Control Systems.* Nauka, Moscow, 1968.
21. KUNTSEVICH V. M. and LYCHAK M. M., *Synthesis of Automatic Control Systems using the Lyapunov Function.* Nauka, Moscow, 1977.

Translated by R.L.Z.